QED in external fields: A functional point of view

J. Alexandre*
Department of Physics, National Technical University of Athens, Zografou Campus, 157 80 Athens, Greece
(Received 26 January 2001; published 26 July 2001)

A functional partial differential equation is set for the proper graphs generating functionals of QED in external electromagnetic fields. This equation leads to the evolution of the proper graphs with the external field amplitude and the external field gauge dependence of the complete fermion propagator and vertex is derived nonperturbatively.

DOI: 10.1103/PhysRevD.64.045011 PACS number(s): 12.20.−m

The study of QED in the presence of external electromagnetic fields started long ago with the computation of the quantum corrections to the Maxwell Lagrangian [1], for which recent and rich studies can be found in [2]. We find also among the studies of the effect of an external field the dynamical chiral symmetry breaking by a magnetic field, the magnetic catalysis [3].

In this framework, we will describe here a non-perturbative approach, similar to the background field methods [4], which will lead us to the external field gauge dependence of the full fermion propagator and the full vertex. For this we will set up a functional differential equation showing the evolution of the effective action (Legendre transform of the connected graphs generator functional) with the amplitude of the external field. The interesting point is that this differential equation is exact and thus independent of perturbative expansions.

The method described here is similar to the one given in [5] where the differential equation described the evolution of the effective action with the mass scale of a scalar theory and led to the well-known one-loop effective action after integration.

We will discuss here the dependence of the proper functions on the gauge of the external electromagnetic field and not of the dynamical one. The gauge dependence with respect to the dynamical field is discussed in [6]. The authors take a usual covariant gauge fixing term and show the dependence of the proper functions on the gauge parameter. Here we will not write this gauge fixing term, although necessary to define the path integral, so as to focus on the effects of the external field. The Lagrangian is

$$\mathcal{L}=-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}+\bar{\Psi}(i\partial\!\!\!/-e\mathcal{A}\!\!\!/-gA\!\!\!/^{ext}-m)\Psi \tag{1}$$

where $\mathcal{A}_\mu$ is the dynamical gauge field of strength $\mathcal{F}_{\mu\nu}$ and A_μ^{ext} the classical (external) electromagnetic field. e is the QED coupling and g the coupling to the external field which is taken different from e so that it can control the amplitude of A_μ^{ext}.

We will introduce the effective action Γ of the model, defined as the Legendre transform of the connected graphs generator functional and our aim will be to derive the exact functional equation

$$\partial_g\Gamma=\mathcal{G}[\Gamma] \tag{2}$$

from which we can extract the evolution of the proper graphs.

In what follows, the Lorentz, Dirac and space-time indices will not be explicitly written if not necessary. We will note ‘‘tr’’ the trace over Dirac indices and ‘‘Tr’’ the trace over Dirac and space-time indices. The computations will be done in dimension d.

The connected graphs generator functional W_g is given by

$$\exp W_g[\bar{\eta},\eta,j]=Z_g[\bar{\eta},\eta,j]=\int\mathcal{D}[\mathcal{A},\bar{\Psi},\Psi]\exp\left\{i\int_x\mathcal{L}+i\int_x(j\mathcal{A}+\bar{\eta}\Psi+\bar{\Psi}\eta)\right\} \tag{3}$$

where $\int_x=\int d^dx$. We note that we do not couple the external field to the source j. W_g has the following functional derivatives:

$$\begin{aligned}\frac{\delta W}{\delta j}&=\frac{1}{Z}\langle i\mathcal{A}\rangle=iA\\ \frac{\delta W}{\delta\bar{\eta}}&=\frac{1}{Z}\langle i\Psi\rangle=i\psi\\ W\frac{\overleftarrow{\delta}}{\delta\eta}&=\frac{1}{Z}\langle i\bar{\Psi}\rangle=i\bar{\psi}\\ \frac{\delta}{\delta\bar{\eta}}W\frac{\overleftarrow{\delta}}{\delta\eta}&=-\bar{\psi}\psi+\frac{1}{Z}\langle\bar{\Psi}\Psi\rangle\end{aligned} \tag{4}$$

where the expectation value $\langle\mathcal{O}\rangle$ of an operator $\mathcal{O}$ is

$$\langle\mathcal{O}\rangle=\int\mathcal{D}[\mathcal{A},\bar{\Psi},\Psi]\mathcal{O}\exp\left\{i\int_x\mathcal{L}+i\int_x(j\mathcal{A}+\bar{\eta}\Psi+\bar{\Psi}\eta)\right\}. \tag{5}$$

*Email address: jalex@central.ntua.gr

Inverting the relations between $(j,\bar{\eta},\eta)$ and $(A,\bar{\psi},\psi)$, we define the effective action $\Gamma_g[A,\bar{\psi},\psi]$ as the Legendre transform of $W_g[j,\bar{\eta},\eta]$ by

$$W_g = i\Gamma_g + i\int_x (jA+\bar{\eta}\psi+\bar{\psi}\eta). \tag{6}$$

From this definition we extract the following functional derivatives:

$$\begin{aligned}
&\frac{\delta\Gamma}{\delta A} = -j\\
&\frac{\delta\Gamma}{\delta\bar{\psi}} = -\eta\\
&\Gamma\frac{\overleftarrow{\delta}}{\delta\psi} = -\bar{\eta}\\
&\frac{\delta}{\delta\bar{\psi}}\Gamma\frac{\overleftarrow{\delta}}{\delta\psi} = -\frac{\delta\bar{\eta}}{\delta\bar{\psi}} = -i\left(\frac{\delta}{\delta\bar{\eta}}W\frac{\overleftarrow{\delta}}{\delta\eta}\right)^{-1}.
\end{aligned} \tag{7}$$

The evolution equation with g of the connected graphs generator functional is, according to Eq. (4)

$$\begin{aligned}
\partial_g W &= \frac{1}{Z}\left\langle -i\int_x \bar{\Psi}\not{A}^{ext}\Psi\right\rangle\\
&= -i\int_x\left(\bar{\psi}\not{A}^{ext}\psi + \not{A}^{ext}\frac{\delta}{\delta\bar{\eta}}W\frac{\overleftarrow{\delta}}{\delta\eta}\right).
\end{aligned} \tag{8}$$

To compute the evolution of Γ_g, one has to keep in mind that the independent variables for this functional are $\bar{\psi},\psi,A$ and g. Taking Eq. (4) into account, we obtain then

$$\begin{aligned}
\partial_g\Gamma &= -i\partial_g W - i\int_x\left(\partial_g\bar{\eta}\frac{\delta W}{\delta\bar{\eta}} + W\frac{\overleftarrow{\delta}}{\delta\eta}\partial_g\eta + \frac{\delta W}{\delta j}\partial_g j\right)\\
&\quad -\int_x(\partial_g jA + \partial_g\bar{\eta}\psi + \bar{\psi}\partial_g\eta) = -i\partial_g W.
\end{aligned} \tag{9}$$

The relation (7) between the second derivatives of W_g and Γ_g implies then the exact functional evolution equation for the effective action:

$$\partial_g\Gamma + \int_x\bar{\psi}\not{A}^{ext}\psi = i\,\mathrm{Tr}\left\{\not{A}^{ext}\left(\frac{\delta}{\delta\bar{\psi}}\Gamma\frac{\overleftarrow{\delta}}{\delta\psi}\right)^{-1}\right\}. \tag{10}$$

We can actually write another evolution equation, which will be more useful, using the equation of motion for the dynamical gauge field A_μ that we now derive again as is done in [7], so as not to miss any contribution.

We can assume that the integral of a derivative vanishes, so that

$$\int\mathcal{D}[\mathcal{A},\bar{\Psi},\Psi]\frac{\delta}{\delta\mathcal{A}_\mu(x)}\exp\left\{i\int\mathcal{L}+i\int(j\mathcal{A}+\bar{\eta}\Psi+\bar{\Psi}\eta)\right\} = 0 \tag{11}$$

which can be written

$$(\Box g^{\mu\nu}-\partial^\mu\partial^\nu)\langle\mathcal{A}_\nu\rangle - e\langle\bar{\Psi}\gamma^\mu\Psi\rangle + Zj^\mu = 0. \tag{12}$$

Using the relations (4) and (7), we find then

$$\begin{aligned}
\frac{\delta\Gamma}{\delta A_\mu} &= (\Box g^{\mu\nu}-\partial^\mu\partial^\nu)A_\nu - e\bar{\psi}\gamma^\mu\psi\\
&\quad + ie\,\mathrm{tr}\left\{\gamma^\mu\left[\frac{\delta}{\delta\bar{\psi}}\Gamma\frac{\overleftarrow{\delta}}{\delta\psi}\right]^{-1}(x,x)\right\}.
\end{aligned} \tag{13}$$

Making the scalar product with A^{ext}, we obtain

$$\begin{aligned}
&\int_x A_\mu^{ext}\frac{\delta\Gamma}{\delta A_\mu} + \frac{1}{2}\int_x F_{\mu\nu}^{ext}F^{\mu\nu} + e\int_x\bar{\psi}\not{A}^{ext}\psi\\
&= ie\,\mathrm{Tr}\left\{\not{A}^{ext}\left(\frac{\delta}{\delta\bar{\psi}}\Gamma\frac{\overleftarrow{\delta}}{\delta\psi}\right)^{-1}\right\}
\end{aligned} \tag{14}$$

up to a surface term. Equations (14) and (10) give then a *linear* evolution equation for the effective action Γ:

$$e\partial_g\Gamma = \int_x A_\mu^{ext}\frac{\delta\Gamma}{\delta A_\mu} + \frac{1}{2}\int_x F_{\mu\nu}^{ext}F^{\mu\nu}. \tag{15}$$

Let us check that $\partial_g\Gamma$ is invariant with respect to an external gauge transformation $A_\mu^{ext}\to A_\mu^{ext}+\partial_\mu\phi$. The second integral of the right hand side of Eq. (15) is of course invariant and the first one becomes after an integration by parts where we disregard the surface term

$$\int_x A_\mu^{ext}\frac{\delta\Gamma}{\delta A_\mu}\to\int_x A_\mu^{ext}\frac{\delta\Gamma}{\delta A_\mu} - \int_x\phi\,\partial_\mu\left(\frac{\delta\Gamma}{\delta A_\mu}\right). \tag{16}$$

But according to Eq. (7), the additional integral vanishes since

$$\partial_\mu\left(\frac{\delta\Gamma}{\delta A_\mu}\right) = -\partial_\mu j^\mu = 0, \tag{17}$$

due to the charge conservation. Thus Eq. (15) is gauge invariant, as expected.

To conclude with the effective action, we can give a general solution of Eq. (15) in terms of the proper graphs generator functional without external field.

If we take the second derivative of Γ with respect to g, we obtain

$$e^2\partial_g^2\Gamma = \int_{xy}A_\mu^{ext}(x)A_\nu^{ext}(y)\frac{\delta^2\Gamma}{\delta A_\mu(x)\delta A_\nu(y)} + \frac{1}{2}\int_x F_{\mu\nu}^{ext}F_{ext}^{\mu\nu} \tag{18}$$

where we again disregarded the surface term. We obtain in general, for $n\geqslant 3$

$$e^n\partial_g^n\Gamma = \int_{x_1\ldots x_n} A^{ext}_{\mu_1}(x_1)\ldots A^{ext}_{\mu_n}(x_n)\frac{\delta^n\Gamma}{\delta A_{\mu_1}(x_1)\ldots\delta A_{\mu_n}(x_n)}. \tag{19}$$

Then we can make the resummation

$$\Gamma_g=\sum_{n=0}^{\infty}\frac{g^n}{n!}\partial_g^n\Gamma_0 \tag{20}$$

and take $g=e$, which leads to the following relation between Γ (with the external field) and Γ_0 (without the external field):

$$\Gamma=\exp\left\{\int_x A^{ext}_\mu(x)\frac{\delta}{\delta A_\mu(x)}\right\}\Gamma_0+\frac{1}{2}\int_x F^{ext}_{\mu\nu}F^{\mu\nu}+\frac{1}{4}\int_x F^{ext}_{\mu\nu}F^{\mu\nu}_{ext}. \tag{21}$$

The integrals involving $F^{ext}_{\mu\nu}$ in Eq. (21) correspond to the subtraction of the kinetic contribution of the external field which does not enter into account in the problem.

We recognize in Eq. (21) the functional translation operator which is the generalization of

$$\exp\left(x_0\frac{d}{dx}\right)f(x)=f(x+x_0) \tag{22}$$

such that we can finally write

$$\Gamma[\bar\psi,\psi,A]=\Gamma_0[\bar\psi,\psi,A+A^{ext}]+\frac{1}{2}\int_x F^{ext}_{\mu\nu}F^{\mu\nu}+\frac{1}{4}\int_x F^{ext}_{\mu\nu}F^{\mu\nu}_{ext}. \tag{23}$$

Thus the effective action of the theory with an external field is the same functional as the one of the theory without external field (but for the bare kinetic term), translated by the vector A^{ext} in the space of the functional variables. This equivalence between the effective action with and without external field is known in the background field methods [4].

Let us now come back to Eq. (15). Its differentiation with respect to the fields leads to the evolution of the proper graphs. Let us take the second derivative with respect to $\bar\psi$ and ψ for vanishing sources. We encounter then the vertex function

$$\Lambda^\mu(z;x,y)=-\frac{1}{e}\frac{\delta}{\delta A_\mu(z)}\frac{\delta}{\delta\bar\psi(x)}\Gamma\frac{\overleftarrow{\delta}}{\delta\psi(y)}\Big|_{A=\bar\psi=\psi=0} \tag{24}$$

and the inverse fermion propagator

$$G^{-1}(x,y)=-i\frac{\delta}{\delta\bar\psi(x)}\Gamma\frac{\overleftarrow{\delta}}{\delta\psi(y)}\Big|_{A=\bar\psi=\psi=0} \tag{25}$$

to obtain the following evolution equation:

$$\partial_g G^{-1}(x,y)=i\int_z A^{ext}_\mu(z)\Lambda^\mu(z;x,y). \tag{26}$$

To fix the idea, we can see that the tree-level proper graphs G^{-1}_{tree} and Λ^μ_{tree} verify Eq. (26) since they are given by

$$G^{-1}_{tree}(x,y)=-i[i\partial\!\!\!/_x-gA\!\!\!/^{ext}(x)-m]\delta(x-y)$$
$$\Lambda^\mu_{tree}(z;x,y)=\gamma^\mu\delta(x-z)\delta(y-z). \tag{27}$$

The evolution equation for the propagator is then obtained by noticing that

$$\partial_g G^{-1}(x,y)=-\int_{z_1z_2}G^{-1}(x,z_1)\partial_g G(z_1,z_2)G^{-1}(z_2,y) \tag{28}$$

and therefore, according to Eq. (26)

$$\partial_g G(x,y)+i\int_{z_1z_2z_3}A^{ext}_\mu(z_1)G(x,z_2)\Lambda^\mu(z_1;z_2,z_3)G(z_3,y)=0. \tag{29}$$

Let us now perform a gauge transformation on the external field $A^{ext}_\mu=A^0_\mu$ and write $A^1_\mu=A^0_\mu+\partial_\mu\phi$ (we will use the same notations for G and Λ^μ corresponding to the two gauges). We obtain after an integration by parts where we omit the surface term

$$\partial_g G_1(x,y)+i\int_{z_1z_2z_3}A^0_\mu(z_1)G_1(x,z_2)\Lambda^\mu_1(z_1;z_2,z_3)G_1(z_3,y)=i\int_{z_1z_2z_3}\phi(z_1)G_1(x,z_2)\partial^{z_1}_\mu\Lambda^\mu_1(z_1;z_2,z_3)G_1(z_3,y). \tag{30}$$

Then the use of the Ward identity

$$\partial^{z_1}_\mu\Lambda^\mu(z_1;z_2,z_3)=\delta(z_2-z_1)G^{-1}(z_1,z_3)-\delta(z_3-z_1)G^{-1}(z_2,z_1) \tag{31}$$

leads us to

$$\partial_g G_1(x,y)-i\Theta_{xy}G_1(x,y)+i\int_{z_1z_2z_3}A^0_\mu(z_1)G_1(x,z_2)\Lambda^\mu_1(z_1;z_2,z_3)G_1(z_3,y)=0 \tag{32}$$

where

$$\Theta_{xy}=\int_x^y dz^\mu[A^1_\mu(z)-A^0_\mu(z)]=\phi(y)-\phi(x). \tag{33}$$

Equation (32) can also be written

$$0=\partial_g\{G_1(x,y)e^{-ig\Theta_{xy}}\}+i\int_{z_1z_2z_3}A^0_\mu(z_1)\{G_1(x,z_2)e^{-ig\Theta_{xz_2}}\}\times\{\Lambda^\mu_1(z_1;z_2,z_3)e^{-ig\Theta_{z_2z_3}}\}\{G_1(z_3,y)e^{-ig\Theta_{z_3y}}\}. \quad (34)$$

We recognize here the equation (29) satisfied by G and Λ^μ in the gauge A^0_μ with the transformation law

$$G_1(x,y)=G_0(x,y)e^{ig\Theta_{xy}}$$
$$\Lambda^\mu_1(z;x,y)=\Lambda^\mu_0(z;x,y)e^{ig\Theta_{xy}} \quad (35)$$

which gives the gauge dependence of the proper graphs G and Λ^μ.

Let us now turn to the constant field case where we know from the work by Schwinger [1] that the bare fermion propagator is of the form

$$G_{tree}(x,y)=e^{ig\int_x^y dz^\mu A^{ext}_\mu(z)}\tilde{G}_{tree}(x-y) \quad (36)$$

where $\tilde{G}_{tree}$ depends on the difference $x-y$ only and is gauge invariant since the gauge dependence of G_{tree} is contained in the phase (36), as can be seen from Eq. (35). With the latter result (35), we can take any gauge and thus will consider that the external potential is linear. We note that in this case the phase can be written

$$g\int_x^y dz^\mu A^{ext}_\mu(z)=\frac{g}{2}(y^\mu-x^\mu)A^{ext}_\mu(x+y). \quad (37)$$

We will check now that the same phase dependence for the complete fermion propagator and vertex is consistent with the differential equation (29). Let us plug

$$G(x,y)=e^{ig\int_x^y dz^\mu A^{ext}_\mu(z)}\tilde{G}(x-y)$$
$$\Lambda^\mu(z;x,y)=e^{ig\int_x^y dz^\mu A^{ext}_\mu(z)}\tilde{\Lambda}^\mu(z-x,z-y) \quad (38)$$

into Eq. (29). We obtain then, after a change of variable and since the potential is linear,

$$\left(i\int_x^y dz^\mu A^{ext}_\mu(z)\right)\tilde{G}(x-y)+\partial_g\tilde{G}(x-y)$$
$$=-i\int_{z_1z_2z_3}A^{ext}_\mu(z_1)\tilde{G}(x-z_2)\times\tilde{\Lambda}^\mu(z_1-z_2,z_1-z_3)\tilde{G}(z_3-y)$$
$$=-i\int_{z_1z_2z_3}A^{ext}_\mu(z_1)\tilde{G}(-z_2)\times\tilde{\Lambda}^\mu(z_1-z_2-u,z_1-z_3+u)\tilde{G}(z_3)$$
$$-iA^{ext}_\mu(v)\int_{z_1z_2z_3}\tilde{G}(-z_2)\times\tilde{\Lambda}^\mu(z_1-z_2-u,z_1-z_3+u)\tilde{G}(z_3) \quad (39)$$

where $u=(x-y)/2$ and $v=(x+y)/2$. We can write after an integration by parts (omitting the surface term)

$$\int_{z_1}\tilde{\Lambda}^\mu(z_1-z_2-u,z_1-z_3+u)=-\int_{z_1}z_1^\mu\partial^{z_1}_\nu\Lambda^\nu(z_1;z_2+u,z_3-u)e^{-ig\int_{z_2+u}^{z_3-u}dz^\mu A^{ext}_\mu(z)} \quad (40)$$

which becomes, with the Ward identity (31),

$$\int_{z_1}\tilde{\Lambda}^\mu(z_1-z_2-u,z_1-z_3+u)=(z_3^\mu-z_2^\mu-2u^\mu)G^{-1}(z_2+u,z_3-u)e^{-ig\int_{z_2+u}^{z_3-u}dz^\mu A^{ext}_\mu(z)}. \quad (41)$$

Taking into account Eq. (41), we simply obtain for the second integral of the right-hand side of Eq. (39)

$$\int_{z_1z_2z_3}\tilde{G}(-z_2)\tilde{\Lambda}^\mu(z_1-z_2-u,z_1-z_3+u)\tilde{G}(z_3)=2u^\mu G(u,-u)=(x^\mu-y^\mu)\tilde{G}(x-y) \quad (42)$$

where we used the fact that $\int_{-u}^u dz^\mu A^{ext}_\mu(z)=0$ since A^{ext} is linear. We see now with the help of Eq. (37) that the terms which do not depend only on the difference $x-y$ cancel in Eq. (39), leading to the consistent $(x-y)$-dependent differential equation (we note $z=x-y$)

$$\partial_g\tilde{G}(z)+i\int_{z_1z_2z_3}A^{ext}_\mu(z_1)\tilde{G}(-z_2)\times\tilde{\Lambda}^\mu\left(z_1-z_2-\frac{z}{2},z_1-z_3+\frac{z}{2}\right)\tilde{G}(z_3)=0. \quad (43)$$

Thus the ansatz (38) verifies the differential equation (29). If we take another gauge which is not linear, the relation (35) will lead us to the same conclusion (38).

To conclude, we stress that the linear functional differential equation (15) leads to general considerations concerning the gauge dependence. This equation expresses the invariance of the functional form of the effective action with respect to a change in the external field amplitude. Its differentiation with respect to the fields leads to relations that have to be satisfied between n-point and $(n+1)$-point proper graphs and, in this respect, these relations play a role similar to Ward identities. More specific information concerning the effects of an external field on the fermion or photon dynamics (dynamical mass generation, photon splitting, electron-positron pair creation, etc.) should be obtained with the help of Eq. (10) and this work is under study.

This work has been done within the TMR project ''Finite temperature phase transitions in particle Physics,'' EU contract number FMRX-CT97-0122. I would like to thank J. Polonyi for introducing me to the functional method described here. I would like to thank also K. Farakos, G. Koutsoumbas, P. Pasipoularides and A. Kehagias for useful discussions, as well as C. Schubert and H. Gies for correspondence.

[1] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936); J. Schwinger, Phys. Rev. **82**, 664 (1951); W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics* (Springer, New York, 1985).

[2] D. Fliegner, M. Reuter, M. G. Schmidt, and C. Schubert, Theor. Math. Phys. **113**, 1442 (1997); C. Shubert, hep-ph/0011331; G. V. Dunne, hep-th/0011036; H. Gies, hep-ph/0010287; V. P. Gusynin and I. A. Shovkovy, J. Math. Phys. **40**, 5406 (1999); P. Pasipoularides, hep-th/0012031.

[3] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Nucl. Phys. **B563**, 361 (1999); E. J. Ferrer and V. de la Incera, Phys. Lett. B **481**, 287 (2000); D.-S. Lee, C. N. Leung, and Y.-J. Ng, Phys. Rev. D **55**, 6504 (1997); V. Skalozub and M. Bordag, Int. J. Mod. Phys. A **15**, 349 (2000); K. Farakos, G. Koutsoumbas, N. E. Mavromatos, and A. Momen, Phys. Rev. D **61**, 045005 (2000).

[4] L. F. Abbott, Nucl. Phys. **B185**, 189 (1981); L. F. Abbott, M. T. Grisaru, and R. K. Schaeffer, *ibid.* **B229**, 372 (1983); R. Ferrari, M. Picariello, and A. Quadri, hep-th/0012090.

[5] J. Alexandre and J. Polonyi, Ann. Phys. (N.Y.) **286**, 1 (2000).

[6] L. D. Landau, A. Abrikosov, and L. Halatnikov, Nuovo Cimento, Suppl. **3**, 80 (1956); C. J. Burden, J. Praschifka, and C. D. Roberts, Phys. Rev. D **46**, 2695 (1992); C. J. Burden and C. D. Roberts, *ibid.* **47**, 5581 (1993).

[7] C. Itzyckson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), Sec. 10.1.1.